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Part I

THE ELEMENTS
OF
THE DIFFERENTIAL CALCULUS,
FOUNDED ON
THE METHOD OF RATES OR FLUXIONS.

PART FIRST.

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1. "I Consider mathematical Quantities in this Place not as consisting of very small Parts; but as describ'd by a continued Motion. Lines are describ'd, and thereby generated not by the Apposition of Parts, but by the continued Motion of Points; Superficies's by the Motion of Lines; Solids by the Motion of Superficies's; Angles by the Rotation of the Sides; Portions of Time by a continual Flux: and so in other Quantities. These Geneses really take Place in the Nature of Things, and are daily seen in the Motion of Bodies. And after this Manner the Ancients, by drawing moveable right Lines along immoveable right Lines, taught the Genesis of Rectangles.

2. "Therefore, considering that Quantities, which increase in equal Times, and by increasing are generated, become greater or less according to the greater or less Velocity with which they increase and are generated; I sought a Method of determining Quantities from the Velocities of the Motions or Increments, with which they are generated; and calling these Velocities of the Motions or Increments *Fluxions*, and the generated Quantities *Fluents*, I fell by degrees upon the Method of Fluxions, which I have made use of here in the Quadrature of Curves, in the Years 1665 and 1666."—*Sir Isaac Newton's Quadrature of Curves.* Stewart's Translation. London: 1745.

"On a reproché à Newton de faire intervenir, sans nécessité, dans ce mode d'exposition, la notion du temps et celle

du mouvement. Le reproche peut être fondé, quant à la notion du mouvement, à laquelle rien n'oblige, en effet, de recourir; mais on devait remarquer que la notion du temps intervient ici par la nature des choses, en raison de ce que le temps est la seule variable essentiellement indépendante, et la seule dont la variation soit essentiellement uniforme, ou la fluxion constante.

"Dans tous les cas, la conception de Newton, appliquée aux grandeurs qui varient effectivement avec le temps, a l'avantage de fixer la signification *réelle* des fonctions dérivées, et par là même de donner à l'avance la raison du rôle qu'elles jouent dans les applications de l'analyse à la discussion des phénomènes physiques. Newton se proposait aussi de fonder la théorie des fonctions sur une idée que l'esprit pût saisir directement, sans passer par la considération des limites et sans s'assujettir à une marche jusqu'à un certain point détournée et indirecte. Il entendait exprimer directement la continuité dans la variation des grandeurs, au moyen du phénomène le plus familier où cette continuité tombe sous les sens. On a objecté, d'après d'Alembert, que, pour *définir* une vitesse continuellement variable, il faut toujours recourir à la considération des limites ; mais, en faisant cette objection, on a mal à propos subordonné la précision des idées à leur définition logique. Un concept existe dans l'entendement, indépendamment de la définition qu'on en donne; et souvent l'idée la plus simple dans l'entendement ne comporte qu'une définition compliquée, quand elle n'échappe pas à toute définition. Tout le monde a une idée directe et exacte de la similitude de deux corps, quoique peu de gens puissent entendre les définitions compliquées que les géomètres ont données de la similitude."—*Théorie des Fonctions et du Calcul Infinitésimal.* Par M. A. A. Cournot. Paris: 1841.

PREFACE.

THE various methods of presenting the Differential Calculus are founded upon four different fundamental conceptions; —first, Infinitesimal Differences; second, Limits; third, Derived Functions; and fourth, Rates or Fluxions. The last, which is due to Sir Isaac Newton, is the one we have adopted; the ordinary notation being, however, retained, and the necessary changes made in the definitions of the symbols. We have furthermore endeavored to establish the differentials of the functions in a manner not only in accordance with these definitions, but independent of the use of limits, infinitesimals, or series. The practicability of this course was first suggested to us by the ingenious treatise of Mr. James Connell, to which we are also indebted for several elementary examples of a kind not usually found in elementary treatises on the Calculus.*

The method of deriving the differentials of functions, exemplified in the case of the square (page 21), and of the logarithmic function (page 39), was first published in the Proceedings of the American Academy of Arts and Sciences; a paper on the method having been communicated to the Society in January, 1873, by Professor J. M. Pierce of Harvard University. In this paper, (which was reprinted from the Proceedings of the Society *) the authors have shown that the method may be extended to all the functions; but for the purposes of elementary instruction it was judged best to employ the usual methods in the cases of the functions whose differentials may be derived directly from those of the Square and the Logarithmic Function; and also to employ another method in the case of the Trigonometric Functions.

---

* *The Elements of the Differential and Integral Calculus, by James Connell.* London: Longman & Co., 1845. (This work is now out of print.)

The mode of teaching the Differential Calculus introduced in these pages has been used by the authors for several years past, in the instruction of their own classes, with unusually satisfactory results. (See Report of the Board of Visitors to the U. S. Naval Academy for 1873.)

The treatise, of which this is the first part, will be completed and published as soon as possible; it is designed to embrace all the topics usually found in text-books on this subject, and to fit the student for the study of treatises in which application is made of the Calculus to physical research.

---

* *On a new Method of obtaining the Differentials of Functions, with especial reference to the Newtonian Conception of Rates or Velocities.* By Professor J. M. Rice and Professor W. W. Johnson (Printed by Welch, Bigelow, & Co., Cambridge, U. S.). A very complete résumé of this paper bv J. W. L. Glaisher of Trinity College, Cambridge, is contained in the *Mathematical Messenger*, for August, 1874.

ANNAPOLIS, MARYLAND,
November, 1874.

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THE

DIFFERENTIAL CALCULUS.

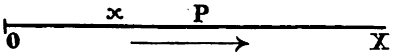
CHAPTER I.

I.

1. In the Differential Calculus, quantities are regarded as undergoing continuous change or variation in magnitude; and the *rates* of these changes, whether of increase or decrease, are treated in connection with the quantities themselves.

RATES.

2. The *rate* of an increasing quantity may be compared to the velocity of a moving point. Thus, if a point P move from a fixed origin O along the straight line OX in the direction of the arrow, the distance OP will be an increasing variable which may be denoted by x, and the *velocity* of the point P will be the *rate* of increase of the varying quantity x.



UNIFORM VELOCITY.

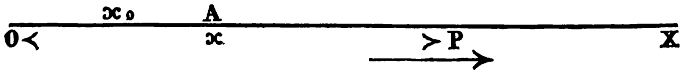
3. The velocity of a point may be uniform or variable; it is *uniform* when the spaces passed over in equal intervals of time are equal, and when, consequently, *the spaces passed over in any intervals of time are proportional to the intervals.*

4. The numerical *measure* of a uniform velocity is the number of units of space passed over in the unit of time; when feet and seconds are taken as the units, respectively, of space and time, the measure of a velocity is the number of feet described per second. If k denote the number of units of space described in one unit of time, and t the number of units of time in any interval, tk will be the measure of the space described in the time t, and regarding t as variable, *this space is proportional to* t.

Again, whenever the spaces described are proportional to their corresponding intervals of time, the velocity is uniform, and the quotient obtained by dividing the number of units of space by the number of units of time occupied in describing this space is constant, and serves as the numerical measure of the velocity.

VARIABLE WITH A UNIFORM RATE.

5. Now let P move with a uniform velocity, whose measure is k, and let t denote the time which has elapsed from a given instant, taken as the origin of time; and let A, in the diagram, be the position of P at that instant. Denote OP, the distance of the moving point from the origin of distances, by x; and denote OA, the value of x corresponding to $t=0$, by x_0. Then AP, the space described in t units of time by the moving point P, contains tk units of space, and we have for OP the formula,



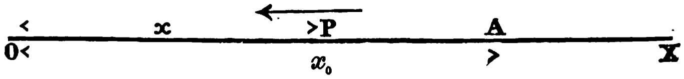
$$x = x_0 + tk, \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

in which x is a varying quantity depending upon the variable t and the constants x_0 and k.

This formula defines a quantity increasing from an *initial value* x_0 with a uniform rate k. The value of x consists of two parts;—first, the initial value; second, the portion generated in the time t.

6. If now the expression for x is of the above form, in which the generated portion is proportional to t, then the quantity x varies uniformly and the coefficient of t is the measure of the rate.

If the point P be moving in the opposite direction with the velocity k, and A still denote the position of P when t is zero, OA will be x_0, and AP will be denoted, as before, by tk, but the formula will become



$$x = x_0 - tk, \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which $-k$ may be regarded as the measure of the rate. This formula for a decreasing quantity may be included in (1) by giving negative values to k, it being understood that a negative rate denotes one of decrease.

VARIABLE VELOCITY.

7. If the velocity of a point be *not* uniform, *its numerical measure at any instant is the number of units of space which would be described in a unit of time, were the velocity to remain constant from and after the given instant.*

Thus, when we speak of a body as having at a given instant a velocity of 32 feet per second, we mean that were the body to continue moving with the same velocity during the whole of the next second, which it had at the instant taken

32 feet would be described; the *actual* space described may be greater or less in consequence of the change of velocity which takes place during the second; it is greater in the case of an increasing velocity, and less when the velocity is decreasing.

8. If it be desired to measure experimentally the velocity attained at any instant by a body in motion, the most obvious method is, when practicable, to give the body from the given instant a continuously uniform velocity identical with that which it has at that instant; the uniform velocity thus attained may then be measured by comparing the space with the time occupied in describing it. This is actually effected in the experimental investigations of the laws of falling bodies by means of Attwood's machine; in these well-known experiments, a variable velocity is converted into a uniform velocity by removing a weight which is the sole cause of the variation.

VARIABLE WITH A VARIABLE RATE.

9. Let us now consider the quantity x generated by the point P, supposed to move with a variable velocity; its rate (being identical with the velocity of P) *will itself be a variable quantity.*

Thus, a little consideration will show that, if $x = t^2$, x will increase as t increases, not uniformly, but with an increasing rate; if, however, $x = \frac{a^2}{t}$, it will decrease as t increases, and with a decreasing rate. Again, if $x = \sin\frac{t\pi}{2}$, it will increase with a decreasing rate, until t becomes unity; then decrease until $t = 3$; the rate of x having its greatest negative value when $t = 2$.

DIFFERENTIALS.

10. The rate of a quantity algebraically expressed is called its *differential*, and is denoted by prefixing a small d to the symbol denoting the quantity.

Thus, if x denote the variable, dx will denote its rate or differential. Marks of parenthesis are used whenever ambiguity is likely to arise from their omission; thus, $d(x^2)$, $d(\sin x)$, $d(xy)$, $d(a^x + x^n)$, &c.

Making use of this notation, we express the conclusions of the preceding article, thus;—if $x = t^2$, dx must be positive, and will itself be an *increasing* quantity; but if $x = \frac{a^2}{t}$, dx will be negative and numerically decreasing; $d\left(\sin\frac{t\pi}{2}\right)$ is positive, and decreasing until $t = 1$, then negative until $t = 3$. These anticipations will be verified when we obtain exact expressions for the values of the differentials.

EXERCISES I.

1. (*a*) What can be said of the rate of a decreasing variable? (*b*) What of the rate of a uniformly increasing variable?

2. (*a*) What can be said of a variable when its differential is positive and increasing? (*b*) What, when it is positive and decreasing? (*c*) What, when it is negative and decreasing?

3. What can be said of the rate of a variable which after ceasing to increase begins to decrease?

4. (*a*) What can be said of a variable when its differential is constant? (*b*) Suppose the value of the constant differential to become zero.

Ans. 2. (*a*) The variable is increasing with an increasing rate.

II.

FUNCTIONS OF AN INDEPENDENT VARIABLE.

11. The examples of variable quantities given above are expressed in terms of the variable t, denoting the time, and are veritably *varying* quantities; that is, such as have definite values at definite *times*, and hence vary with a rate which has a definite value at each instant. The term variable is however applied also to quantities which are merely susceptible of a variety of values, and of a continuous change in value, like the abscissa and ordinate of a point moving in a curve.

The variables which present themselves when mathematical principles are applied to the solution of problems, are usually so connected that one may be expressed in terms of another, as in the case mentioned above of the ordinate and abscissa in a curve whose equation is known.

12. Any analytical expression, whose value depends in any way upon that of a quantity involved in it, is said to be a *function* of that quantity.

Thus, $m+(n-x)x$, $\dfrac{x^4-ax+b}{\sqrt{x^2-1}}$, $\log(x+a)$, $\tan x$, a^x, &c., are functions of x.

But $ax^2-(bc+ax^2)$, $\log\left[\dfrac{(a^3-ax^2)(a-b)}{(x-a)(x+a)m}\right]$, and $y[x^2+(n-x)x]-nxy$ are *not* functions of x, since they do not depend upon it for their values (x disappearing from each expression when it is reduced to its simplest form).

The term function has already been rendered somewhat familiar by its occasional use in Trigonometry and in Analytical Geometry. In the Calculus, however, it plays a much more important part, being used with great frequency, and with a signification so extended as to embrace not only quantities which can be represented by Algebraic, Trigonometric, and Logarithmic expressions, but in some cases those which are not susceptible of mathematical definition.

13. It is customary to represent the value of a function by a single letter;—thus, $y = (a - x) \tan x$, $y = \log x$, and $y = ax^n - bx^2 + c$; in each of these expressions, y is said to be a function of x.

It is often important to express that y is a function of x without specifying the *form* of the function; in such cases we write $y = f(x)$. If more than one function of x is under consideration the expressions $f(x)$, $F(x)$, $\varphi(x)$, &c., are used.

CLASSIFICATION OF FUNCTIONS.

14. In an equation expressing the relation between the variables x and y, either variable may be regarded as a function of the other, the latter is then called the *independent* variable. If one variable is expressed *directly* in terms of the other, the former is said to be an *explicit* function of the latter; thus,—in the above examples, y is an explicit function of x. When an equation involving x and y is not of this form, y is said to be an *implicit* function of x.

Explicit functions are divided into *Algebraic* and *Transcendental* functions; the latter class includes Exponential functions, as a^x, $xa^{\sqrt{x+x^2}}$; Logarithmic functions as $\log x$, $(a + x) \log \sqrt{x^n - a^n}$, and Trigonometric functions as $\sin x$, $\tan x$, $(a^2 + x^2) \sec x^2$.

EXAMPLES II.

1. Is $\dfrac{a+x}{2} + \dfrac{a-x}{2}$ a function of x?

2. Is $\dfrac{x + \dfrac{ax - x^2}{a+x}}{1 - \dfrac{a-x}{a+x}}$ a function of x? Is it a function of a?

3. Of what quantities is $(x+a-y)^2 - a(a+2x) + x(2y-x)$ a function?

4. Given, $\dfrac{1}{y} + \dfrac{1}{x} - \dfrac{1}{a} = 0$, to make y an explicit function of x, and x an explicit function of y.

5. What explicit functions of x are included in the implicit function $y^2 - 2ay + a^2 - x^2 = 0$?

6. Show that $\frac{\log y}{\log x} = n$ is equivalent to the explicit form $y = x^n$.

7. In the curve whose equation is $xy + x + y + 2 = 0$, what function of the ordinate is the abscissa?

8. Given the equations,—

$$n + 1 = n(\cos^2\theta' + \cos\theta'\cos\theta + \cos^2\theta),$$

and

$$n - 1 = n(\sin^2\theta' + \sin\theta'\sin\theta + \sin^2\theta),$$

eliminate n, and make θ' an explicit function of θ.

9. If $f(x) = x^2$, what does $f(x + h)$ equal? $f(z)$? $f(2x)$? $f(x^2)$? $f(x^2 - x)$? $f(1)$? $f(10)$?

10. If $F(x) = a^x$, what does $F(a)$ equal? $F(1) - F(0)$? Show that $[F(x)]^2 = F(2x)$.

11. If $f(\theta)=\sin\theta$, what does $f(0)$ equal? $f\left(\frac{\pi}{6}\right)$? $f\left(\frac{\pi}{2}\right)$? $f(\pi)$?

12. If $\varphi(x) = mx$, show that $\varphi(x) + \varphi(y) = \varphi(x + y)$.

13. If $\varphi(x) = x^n$, show that $\varphi(x).\varphi(y) = \varphi(xy)$.

14. If $\varphi(x) = \log x$, show that $\varphi(x) + \varphi(y) = \varphi(xy)$.

15. If $\varphi(x) = a^x$, show that $\varphi(x).\varphi(y) = \varphi(x + y)$.

16. If the expression $xf(x)$ is not a function of x, show what must be the form of the function $f(x)$, and prove that that $xf(x) = zf(z)$.

Ans. $f(x) = \frac{c}{x}$, where c is an unknown constant.

17. Conversely, if $xf(x) = zf(z)$, what must be the form of $f(x)$?

18. If $\frac{1}{x}.f(x) = \frac{1}{z}.f(z)$, what must be the form of $f(x)$?

---

5. $y = a \pm x$.

8. By adding and reducing we obtain $\theta' = \theta \pm \frac{\pi}{2}$.

III.

DIFFERENTIALS OF FUNCTIONS.

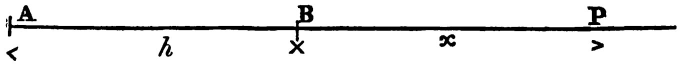
15. When a quantity and its function are regarded as undergoing continuous change, it is obvious that the rate of change of the function will depend upon the rate of change in the variable; that is, the differential of a function of x will depend upon dx. We now proceed to express the differential of each of the simple functions in terms of the differential of the independent variable. The value of dx is to be regarded as entirely arbitrary. The following propositions are immediate deductions from the method of measuring rates explained in the articles on variable rates.

Proposition I.

16. *If* h *denote a constant quantity, and* x *a variable quantity, the rate of* x + h *will equal that of* x, *or*

$$d(x+h) = dx.$$

For if h be represented by AB and x by BP, A and B



being fixed points and P a moving point, the *rate* of P, whether uniform or variable, will belong equally to BP (or x) and AP (or $x + h$)

hence $$d(x+h) = dx.$$

Proposition II.

17. *If* m *denote a constant factor, the rate of the function* mx *will equal the product obtained by multiplying the rate of* x *by* m, *or*

$$d(mx) = mdx.$$

If x varies with a uniform rate, it can be expressed thus (Art. 5),—

$$x = x_0 + tk.$$

Whence $$mx = mx_0 + mtk,$$

in which the generated portion is proportional to the time; hence (Art. 6) mx varies uniformly, and the coefficient (mk) of t is the measure of the rate; in this case, therefore,

$$d(mx) = mdx.$$

Again, if x varies with a variable rate, the proposition still holds; for, if at any instant k is the measure of the rate, or portion which would be generated in a unit of time were the rate to remain uniform (Art. 7 and Art. 9), then as before mtk would be the generated portion of mx in t units of time, and mk generated in one unit of time would be the measure of the variable rate; hence in general

$$d(mx) = mdx.$$

The above demonstration is applicable when the quantity denoted by m is negative;

hence $$d(-mx) = -mdx;$$

also if $m = -1$, $$d(-x) = -dx.$$

PROPOSITION III.

18. If x *and* y *denote two variables, the rate of* x+y *is the sum of the rates of* x *and* y.

Let x_0 denote a particular value of x, and k that of dx corresponding to it; let y_0 and k' denote the simultaneous values of y and dy respectively.

Then, if the rates of x and y were to remain constant, their values at the end of the time t would be

$$x = x_0 + tk,$$

and $$y = y_0 + tk';$$

hence that of $x + y$ would be

$$x + y = x_0 + y_0 + t(k + k');$$

that is, the rate of $x + y$ would also remain constant and equal to $k + k'$,

hence $$d\,(x + y) = dx + dy.$$

As this result is true for any values of dx and dy, it is true when they are variables.

19. To illustrate this proposition let us suppose a man walking upon the deck of a ship, and in the direction of the ship's motion. Let the distance of the stern-post of the ship from a fixed point in the wake of the ship be denoted by the variable x, and let the distance of the man from the stern-post be denoted by y; then dx is the velocity of the ship and dy the velocity of the man with reference to the deck of the ship. Now $x + y$ will denote the distance of the man from the fixed point, and $d\,(x + y)$ his actual velocity with reference to it, then the proposition

$$d\,(x + y) = dx + dy$$

indicates that the actual velocity of the man with reference to a fixed point equals the velocity of the ship plus that of the man with reference to the ship's deck.

EXAMPLES III.

1. Find by Prop. II. the differential of $\frac{x}{25}$, and of $\frac{x}{a-2}$.

Ans. $\frac{1}{25}dx$, and $\frac{1}{a-2}dx$.

2. Find the differential of $\frac{a-x}{m^2}$, and of $\frac{x-a}{m^2}$. [Notice carefully the constant parts.]

3. Find the differential of $\frac{a+b+(a-b)\,x}{a^2-b^2}$.

Ans. $\frac{dx}{a+b}$.

4. Find the differential of $\frac{a+x}{a+b}$, and of $\frac{b\,(x+y)}{a\,(a+b)}$.

5. Given $2x - 3y = 5$, to differentiate y as a function of x.

6. Given $x + y = c$, to find dy in terms of dx. What does the result express when $c = 0$?

7. If $dy = dx$, what can be inferred of the relation between x and y?

8. If $y = \frac{a + bx}{c}$, when is it an increasing, and when is it a decreasing function of x?

9. Given $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$, to express dz in terms of dx and dy.

10. A man whose height is 6 feet walks from under a lamp-post at the rate of 3 miles an hour. At what rate is the extremity of his shadow travelling, supposing the light to be 10 feet above the level pavement on which he is walking?

[Draw a figure, and denote the variable distance of the man from the lamp-post by x, and the distance of the extremity of his shadow from the post by y].

Ans. 7½ miles per hour.

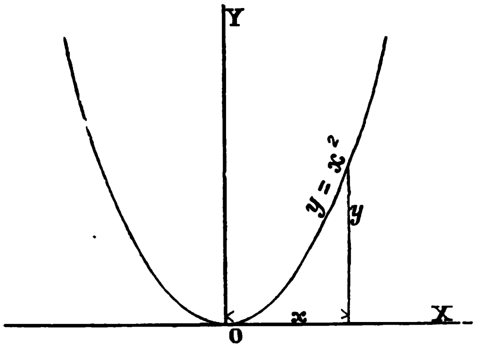
11. At what rate does the man's shadow [Ex. 10] increase in length?

IV.

FUNCTIONS ILLUSTRATED BY CURVES.

20. If y denote a function of x, and a curve be constructed, x being an abscissa and y the corresponding ordinate, this curve will serve to illustrate the variation of the function y when x is supposed to vary uniformly.

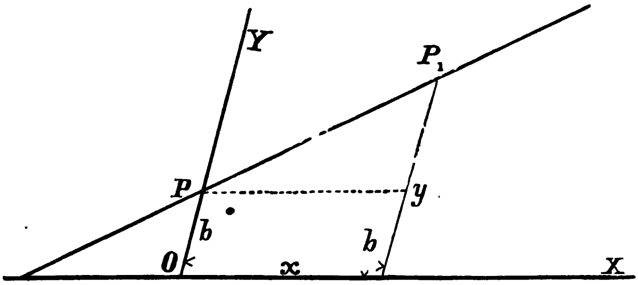
The figure presents the curve corresponding to the function $y = x^2$. If this curve be generated by a point of which the abscissa, starting from the value zero, increases uniformly (in other words, if dx be constant), the rate of increase of the ordinate is evidently variable, and it is likewise evi-



dent that the motion of the generating point in the direction of the axis of y is an accelerated motion; dy therefore denotes in this case an increasing variable rate.

UNIFORM RATE OF BOTH VARIABLES.

21. Let the ordinate y move along OX (keeping always parallel to OY) with a uniform rate or velocity dx, and, at



the same time, let it increase in length with a uniform rate dy. Let P_1 denote the position of the point at the end of t units of time. Since dx is constant, and since x starts with the initial value zero, we have (Art. 5)

$$x = tdx,$$

and since dy is constant, while y starts with the initial value b,

$$y = b + tdy.$$

Eliminating t $$y = b + x\frac{dy}{dx};$$

dy and dx being both constants, we put m for $\frac{dy}{dx}$, thus—

$$y = mx + b.$$

This result, being independent of any particular value of t, is true for any point of the path, and is therefore its equation; moreover, it is obviously the equation of a straight line, since m is a constant.

It therefore follows, that *a point moving with uniform rates in two directions will describe a straight line;* the line thus described is evidently the diagonal of a parallelogram of which the sides are tdx, and tdy.

VARIABLE RATES WITH A CONSTANT RATIO.

22. Evidently, the above line may be described by a point moving with a variable velocity; in which case, the velocities in the directions of OX and OY will be variable; but the *ratio* of these velocities will still be constant; for, from the equation of the path, by taking differentials of both members of

$$y = mx + b,$$

we have $$dy = mdx,$$

or $$\frac{dy}{dx} = m.$$

The ratio m evidently determines the *direction* of the straight line, or, what is the same thing, the direction of the motion of the generating point. When the axes are rectangular, m is the tangent of the inclination of the line to the axis of x.

23. Since an equation of the form

$$y = mx + b$$

is represented by a straight line, functions of the form $mx + b$ are called *linear functions.* In an equation of the first degree ($Ax + By + C = 0$), each variable is a linear function of the other; equations of this form are therefore called *linear equations.*

RATES WITH A VARIABLE RATIO.

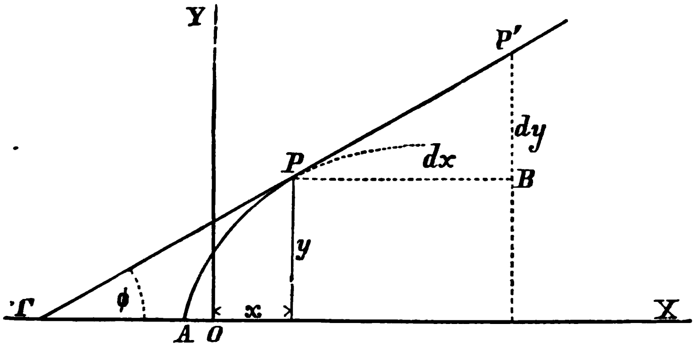
24. If, while the rate dx is constant, dy is variable, the point will describe a curve, as in the case of the function $y = x^2$, to which the corresponding curve is traced above. It is further evident that the value of $\frac{dy}{dx}$ determines the direction of the motion of the generating point; because, from what has been shown above, when this ratio is constant, the point moves in a constant direction, and when it undergoes continual change the point moves in a variable direction, or in other words describes a curve.

THE RATIO OF DIFFERENTIALS ILLUSTRATED GEOMETRICALLY.

25. If, when the generating point of the curve AP, in the figure, arrives at a given position P, the rate dy as well as dx becomes constant, the point will thereafter retain a constant direction and rate of motion, and will describe the straight line PP', which has the direction of the motion of the generating point of the curve when at the position P; in accordance with the following definition, this line is called a *tangent.*

The tangent to a curve at any point is the straight line which passes through the point, and has the direction of the curve at that point.

This general definition of a tangent line will be shown hereafter to give identical results with those already familiar to the student, in the cases of the circle and of the other conic sections.



26. If (the co-ordinate axes being rectangular) we denote the inclination of this tangent line to the axis of x by φ, we shall have $\varphi = P'PB$ (see diagram); hence

$$\frac{dy}{dx} = \tan\varphi;$$

in which $\tan\varphi$ has a definite value for each point of the curve, but varies from point to point, and is therefore capable of being expressed as a function of x. Hence when the curve illustrating y as a function of x is constructed, the value of the ratio $\frac{dy}{dx}$ is represented by the trigonometric tangent of the inclination to the axis of x of the motion of the generating point, the direction of this motion being one or the other of the two opposite directions on the tangent line through the point. The angle φ at any point may have either of two values differing by 180°, but

$$\tan\varphi = \tan(\varphi \pm 180°);$$

hence the value of $\tan\varphi$ at any point of the curve is perfectly definite.

FUNDAMENTAL THEOREM.

27. Since the path of the generating point of the curve, illustrating an explicit function of x is determined by the form of the function, in other words by the equation of the curve, its *direction* at any point, and consequently the value of $\tan \varphi$, does not depend upon the value of dx; *therefore the value of* $\frac{dy}{dx}$ *does not depend upon* dx, *but is either constant as in the case of the linear function, or may be expressed simply as a function of* x.

28. Thus, since we know that the equation

$$y = ax^5$$

does not represent a straight line, we know also that $\frac{dy}{dx}$ cannot in this case be constant, and therefore by the theorem it is a function of x; we shall see hereafter that it is in fact $5ax^4$. In general, if $y = f(x)$,

$$\frac{dy}{dx} = f'(x),$$

some other function of x; we may therefore write

$$dy = f'(x)\, dx.$$

The function $f'(x)$ is called the *derivative* of the function $f(x)$. When $f(x)$ is the ordinate of a curve, $f'(x) = \tan \varphi$.

EXAMPLES IV.

1. Trace the curve $y = \log x$. Is $\tan \varphi$ in this case an increasing or a decreasing function of x?

2. Given, the function $y = \frac{a^2}{x}$; if dx is constant and positive, what can be said of dy? What of dx, if dy is constant and positive?

3. Determine, by tracing the curve corresponding to $y = a^x$, the character of the function $\frac{dy}{dx}$.

4. Trace the curve $y = x^3$. Is $f'(x)$ ever negative for this curve? Is $f'(x)$ on the left of the origin an increasing or a decreasing function of x? On the right of the origin?

5. Trace the curve $y^2 = x^3$: determine the sign of $d\,(\tan\varphi)$ for the branch above the axis of x; also for the branch below the axis of x.

6. If $\frac{dy}{dx} = x^2$, can y ever be a decreasing function of x?

7. If $\frac{dy}{dx} = x$, when is y an increasing function of x? When is it a decreasing function of x?

8. If $f(x) = \sin x$, does $f'(x)$ ever change sign?
If $f(x) = \tan x$, does $f'(x)$ ever change sign?

9. If a point move so that the rate of its abscissa is twice that of its ordinate (both positive), and cross the axis of y two inches below the origin, what is the equation of its path?
Ans. $y = \frac{1}{2}x - 2$.

10. If a point move in the straight line $2y - 7x - 5 = 0$ so that its ordinate decreases at the rate of 3 units per second, how is it moving in the direction of the axis of x?

11. If a point moving in a curve passes through the point (5, 3) moving at equal rates upward and towards the left, what is the equation of the tangent line to the curve at that point?

12. If a point is moving in the straight line

$$x \cos\alpha + y \sin\alpha = p,$$

and if its rate in the positive direction of the axis of x is $l \sin\alpha$, what is its rate of motion in the direction of the axis of y?
Ans. $-l \cos\alpha$.

V.

DIFFERENTIATION.

29. *To differentiate an equation* is to express the equality of the differentials of its members. This equality evidently exists when the two members are constants, as, for example, if we have $x^2 + ay^2 + by + c = 0$; and also when they are functions which are equal for all values of the variables upon which they depend.

In such cases, therefore, we are able from a given equation to derive another by differentiation; thus, from the identical equation

$$(x - a)(x + a) = x^2 - a^2,$$

we derive
$$d[(x - a)(x + a)] = d(x^2).$$

Again, the relation between x and y being expressed by the equation

$$2x - 3y = 5,$$

we obtain
$$2dx - 3dy = 0.$$

POLYNOMIALS.

30. The three elementary propositions given above show: that, in obtaining the differential of a polynomial, the differential of each of its terms is to be taken separately; that a constant factor of a term whether positive or negative appears simply as a coefficient of the differential of the variable factor of the term; and that the differential of a constant term of a polynomial is always zero; thus,—

$$d(x^3 + 3x^2 - 2xy - \frac{x}{2} + 5) = d(x^3) + 3d(x^2) - 2d(xy) - \tfrac{1}{2}\,dx.$$

We have still to determine the differentials of the variable factors.

COMPLEX FUNCTIONS.

31. *To differentiate a function* is to express its differential in terms of the differential of the variable upon which the value of the function depends. Thus, to differentiate x^2 is to express $d(x^2)$ in terms of dx, and to differentiate $\sin x$ is to express $d(\sin x)$ in terms of dx. To differentiate $\sin^2 x$, we have first to express $d(\sin^2 x)$ in terms of $d(\sin x)$, and then, by substituting the value of the latter differential, we finally express $d(\sin^2 x)$ in terms of dx.

We have now, therefore, only to establish formulas for the differentiation of the simple functions.

EXAMPLES V.

1. Prove that,—

$$d(x+h)^2 = d\,(x^2) + 2h dx;$$

also, that—

$$d(x-y)^2 = d(x^2) - 2d(xy) + d(y^2).$$

2. Given $ay + bx + 2cx + a + 8 = 0$, to find $\frac{dy}{dx}$.

3. Given $f(x) = ax + 8x - cx + x + b$, to find $f'(x)$.

4. Given $y \log a + x \sin \alpha - y \cos \alpha - ax + \tan \alpha = 0$, to find $\frac{dy}{dx}$.

5. Given $ay \sin \alpha - ax + ax \cos \alpha - b^2 \sec \alpha = 0$; show that φ is constant and equal to $\frac{1}{2}\,\alpha$.

CHAPTER II.

THE DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

VI.

THE SQUARE (x^2).

32. In establishing the formulas for the differentiation of the simple algebraic functions of an independent variable, we find it most convenient to begin with the square; the object of this article is therefore to express $d(x^2)$ in terms of x and dx.

We first deduce a relation between two values of the derivative of the function and the corresponding values of the independent variable; for this purpose we assume two values of the variable having a constant ratio m, thus:

$$z = mx, \quad \text{then will } dz = m\,dx \,\ldots\ldots\ldots\ldots (1)$$

and

$$z^2 = m^2x^2, \quad \therefore \quad d(z^2) = m^2 d(x^2) \ldots\ldots\ldots\ldots (2)$$

Dividing equation (2) by equation (1),

$$\frac{d(z^2)}{dz} = m\frac{d(x^2)}{dx}, \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

a relation between two values of the derivative; putting for m its value $\frac{z}{x}$, we obtain

$$\frac{d(z^2)}{dz} = \frac{z}{x} \cdot \frac{d(x^2)}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

the relation required. Dividing by z to *separate* the variables, we have

$$\frac{1}{z} \cdot \frac{d(z^2)}{dz} = \frac{1}{x} \cdot \frac{d(x^2)}{dx} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

This equation is true for all values of the quantities x, z,

dx and dz; for, by the Fundamental Theorem $\frac{d(z^2)}{dz}$ is independent of dz, and $\frac{d(x^2)}{dx}$ is independent of dx, the derivatives being functions respectively of z and of x simply; moreover, the equality exists independently of any particular value of m, since it has been eliminated.

The first member of this equation being independent of the value of dz can be a function of no variable quantity except z, and so likewise the second member can be a function of no variable quantity except x.

If, therefore, we denote x^2 by $f(x)$, and adopt the notation of Art. 28 for the derivative, we may write equation (5) thus:

$$\frac{1}{z}.f'(z) = \frac{1}{x}.f'(x) \ldots\ldots\ldots\ldots\ldots (5')$$

Now, since the form of the first member of this equation is the same as that of the second member, it follows that the value of the expression $\frac{1}{x}.\ f'(x)$ or $\frac{1}{x}.\frac{d(x^2)}{dx}$ does not change when x is changed to z, the latter representing, it must be remembered, *any other* value of the independent variable; the value of this expression is therefore constant, and denoting it by c we write

$$\frac{1}{x}.\frac{d(x^2)}{dx} = c; \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

whence $$d(x^2) = cxdx \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

To determine the unknown constant c we apply this result to the identity

$$(x+h)^2 = x^2 + 2hx + h^2; \ldots\ldots\ldots\ldots (8)$$

differentiating each member (Art. 29) by formula (7) we have

$$c(x+h)d(x+h) = cxdx + 2hdx; \ldots\ldots\ldots (9)$$

since $d(x+h) = dx$, equation (9) reduces to

$$chdx = 2hdx$$

or $$(c-2)\ hdx = 0;$$

and, since h and dx are arbitrary quantities, we have

$$c = 2,$$

which substituted in (7) gives

$$d(x^2) = 2xdx.\dots\dots\dots\dots\dots\dots(a)$$

That is, *the differential of the square of a variable equals twice the product of the variable and its differential.*

33. Thus, if a variable is increasing at the rate of 5 units per second when the value of the variable is 7 units, the rate of its square is $2 \times 7 \times 5 = 70$; that is, the square is increasing at the rate of 70 units per second.

Since x and dx are arbitrary, we may substitute for them any variable and its differential. Equation (a) therefore enables us to differentiate the square of any variable whose differential is known, thus—

$$d(2x - 3)^2 = 2(2x - 3)2dx;$$

and

$$\begin{aligned} d(ax^2 + bx)^2 &= 2(ax^2 + bx)d(ax^2 + bx) \\ &= 2(ax^2 + bx)(2ax + b)dx. \end{aligned}$$

THE SQUARE ROOT ($\sqrt{x}$).

34. We shall now proceed to derive the differential of the *square root.* The square and the square root are both particular cases of the power (x^n), which will be differentiated hereafter.

Put $\quad y = \sqrt{x},$

whence $\quad y^2 = x,$

differentiating by (a) $\quad 2ydy = dx,$

or $\quad dy = \dfrac{dx}{2y};$

but $y = \sqrt{x}$, $\therefore$ $\quad d(\sqrt{x}) = \dfrac{dx}{2\sqrt{x}}.\dots\dots\dots\dots\dots(b)$

That is, *the differential of the square root of a variable* *equals the quotient arising from dividing the differential of the variable by twice the radical.*

Thus,

$$d\sqrt{a^2 - x^2} = \frac{-x\,dx}{\sqrt{a^2 - x^2}}.$$

If $a = 13$, and if x be increasing at the rate of 2 units per second when it reaches the value 5, this result shows that $\sqrt{a^2 - x^2}$ is then decreasing at the rate of $\frac{5}{6}$ of a unit per second.

Examples VI.

1. Differentiate $(2x - 3)^2$, and find the numerical value of its rate, when x has the value 8, and is decreasing at the rate of one unit per second.

2. Differentiate $\left(\frac{x}{a}\right)^2$ and $\sqrt{\frac{x}{a}}$. [In the first case $\frac{1}{a^2}$, and in the second $\frac{1}{\sqrt{a}}$ is a constant factor of the variable part of the function.]

3. Does x^2 ever decrease while x increases?

4. How does $\sqrt{x}$ vary when x increases uniformly?

5. Derive the differential of $-\sqrt{x}$ by the method employed in Art. 34, assuming $y = -\sqrt{x}$, &c.

6. Differentiate $(x^2 + ax)^2$, and $\sqrt{\frac{x^2 + 2ax}{a^2 + 2ab}}$.

7. Differentiate the identity $(\sqrt{x} + \sqrt{a})^2 = x + a + 2\sqrt{ax}$.

8. How does $x - \sqrt{x^2 - a^2}$ vary when x increases uniformly?

9. Differentiate $\sqrt{a + bx + cx^2}$, and $\sqrt{y^2 + x^2}$.

10. Differentiate $\sqrt{x + \sqrt{1 + x^2}}$.

11. A man is walking on a straight path at the rate of 5 ft. per second; how fast is he approaching a point 120 ft. from the path in a perpendicular, when he is 50 ft. from the foot of the perpendicular?

Let x denote the variable distance of the man from the foot of the perpendicular, and let a denote the length of the perpendicular (120 ft.); then the distance of the man from the point is $\sqrt{a^2+x^2}$, of which the rate of change is denoted by $d(\sqrt{a^2+x^2}) = \frac{xdx}{\sqrt{a^2+x^2}}$. At the instant considered, $x =$ 50 ft. while $a =$ 120 ft., and $dx = -$ 5 ft. per second (since the man is *approaching* the foot of the perpendicular at that rate).

By substitution we have $\frac{xdx}{\sqrt{a^2+x^2}} = -1\frac{12}{13}$; hence his distance from the point is diminishing (that is, he is approaching it) at the rate of $1\frac{12}{13}$ ft. per second.

12. If the side of an equilateral triangle increase uniformly at the rate of 3 ft. per second, at what rate is the area increasing, when the side is 10 ft.? *Ans.* $15\sqrt{3}$.

13. A stone is dropped into still water and produces a series of continually enlarging concentric circles; it is required to find the rate at which the area of one of them is enlarging, when its diameter is 12 inches, supposing the wave to be then receding from the centre at the rate of 3 inches per second. *Ans.* 36π.

14. If a circular disk of metal expand by heat so that the area of each of its faces increases at the rate of 0.01 sq. ft. per second, at what rate is its diameter increasing?

15. A man standing on the edge of a wharf is hauling in a rope attached to a boat at the rate of 4 ft. per second. The man's hands being 9 ft. above the point of attachment of the rope, how fast is the boat approaching the wharf when she is at a distance of 12 ft. from it?

VII.

THE PRODUCT (xy).

35. If x and y denote any two variables, xy is a function of both, and its differential depends upon x, y, dx, and dy.

In order to derive this differential, we express xy by means of squares, since we have already obtained a formula for the differentiation of the square. Thus, from the identity

$$(x + y)^2 = x^2 + 2xy + y^2,$$

we derive $$xy = \tfrac{1}{2}(x + y)^2 - \tfrac{1}{2}x^2 - \tfrac{1}{2}y^2.$$

Differentiating, $$d(xy) = (x + y)(dx + dy) - xdx - ydy,$$

$$\therefore \qquad d(xy) = ydx + xdy. \dots\dots\dots\dots\dots (c)$$

That is, *the differential of the product of two variables equals the sum of the products obtained by multiplying each variable by the differential of the other.*

Since x and y denote any variables whatever, and dx and dy their differentials, we can substitute for x and y any variable expressions, and for dx and dy their corresponding differentials. Thus—

$$d[(1 + x^2)\sqrt{a^2 - x^2}\,] = \sqrt{a^2 - x^2}.\, 2xdx - \frac{(1 + x^2)xdx}{\sqrt{a^2 - x^2}}$$

$$= \frac{2a^2 - 3x^2 - 1}{\sqrt{a^2 - x^2}}xdx.$$

THE RECIPROCAL $\left(\frac{1}{x}\right)$.

36. The reciprocal is a function of x whose differential may now be obtained by means of its implicit form.

Denoting the function by y we have

$$y = \frac{1}{x} \quad \therefore \quad xy = 1.$$

Differentiating this equation by formula (*c*) we obtain

$$ydx + xdy = 0,$$

whence $$dy = -\frac{ydx}{x};$$

substituting the value of y,

$$d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}. \quad \ldots\ldots\ldots\ldots\ldots (d)$$

That is, *the differential of the reciprocal of a variable is equal to the negative of the quotient obtained by dividing the differential of the variable by the square of the variable.*

Formula (*d*) enables us to differentiate any fraction of which the denominator alone is variable; thus—

$$d\left(\frac{a+b}{a+x}\right) = -(a+b)\frac{dx}{(a+x)^2}.$$

THE QUOTIENT $\left(\frac{x}{y}\right)$.

37. By the term *quotient* as used in this article, we mean a fraction whose numerator and denominator are both variables. In deriving its differential, the quotient is regarded as the product of its numerator by the reciprocal of its denominator. Thus, applying formulas (*c*) and (*d*),

$$d\left(\frac{x}{y}\right) = d\left(x . \frac{1}{y}\right) = \frac{1}{y}dx + xd\left(\frac{1}{y}\right)$$

$$= \frac{dx}{y} - \frac{xdy}{y^2},$$

$$\therefore \quad d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2} \quad \ldots\ldots\ldots\ldots\ldots (e)$$

That is, *the differential of a quotient is found by subtracting the product of the numerator by the differential of the denominator from the product of the denominator by the differential of the numerator, and dividing the difference by the square of the denominator.*

It will be noticed that the negative sign is attached to the part dependent upon the differential of the denominator.

This formula is to be used only when both terms of the fraction are variable; for when the *numerator is constant*, the fraction is equivalent to the product of a constant and the reciprocal of a variable, and when the *denominator is constant*, to the product of a constant by a variable factor.

EXAMPLES VII.

1. Show from formula (*d*) that the reciprocal is always a decreasing function. When is its rate most rapid?

2. If the sides of a rectangle increase uniformly, the one at the rate of 2 inches per second, and the other at the rate of 3 inches per second; what is the rate of increase of the rectangle, when its sides are 8 inches and 6 inches respectively?

3. What is the rate of this rectangle at the instant it becomes a square? [To find the common value of the sides at this instant, express each by means of the formula of Art. 5.]

Ans. 60 square inches.

4. Differentiate the identity $(x+y)(x-y) = x^2 - y^2$.

5. Differentiate the identity $\frac{a}{x} + \frac{x}{a} = \frac{a^2+x^2}{ax}$.

6. From the identity $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$ derive the formula for differentiating the product.

7. Differentiate $\frac{a-x}{\sqrt{x}}$ and $\frac{\sqrt{x^2-a^2}}{x}$.

8. Differentiate $\frac{a+bx+cx^2}{x}$. [Notice the advantage of using the form $\frac{a}{x} + b + cx$.]

9. Differentiate the identity $\frac{2a^2-x^2}{a-x} = \frac{a^2}{a-x} + a + x$.

10. Deduce a formula for differentiating the fourth power,—writing x^4 in the form $x^2 . x^2$; also in the form $(x^2)^2$.

11. Differentiate $\frac{a^2-b^2}{a^2-x^2}$; also $\frac{r^2+x^2}{2rx}$.

12. Differentiate $\frac{ab}{cx\sqrt{x^2-a^2}}$. $\left[\text{Use the form } \frac{ab}{c} . \frac{1}{x\sqrt{x^2-a^2}}.\right]$

13. Differentiate $\frac{a^2-b^2}{\sqrt{x}\sqrt{2a-x}}$. $\left[=(a^2-b^2)\frac{1}{\sqrt{2ax-x^2}}.\right]$

14. Differentiate $\frac{bx}{\sqrt{2ax-x^2}}$.

15. Differentiate $\frac{1}{\sqrt{1+x}}+\frac{1}{\sqrt{1-x}}$.

16. Differentiate $\frac{x}{\sqrt{1-x^2}}$.

17. Differentiate $\sqrt{\frac{1+x}{1-x}}$.

18. Differentiate $\frac{x}{\sqrt{a^2+x^2}-x}$. [Free the denominator from radicals before differentiating.]

19. Differentiate $\sqrt{x+\sqrt{1+x^2}}$.

Ans. $\frac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}}\,dx$.

20. Differentiate $\sqrt{x^2+a\sqrt{a^2+x^2}}$.

21. Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, to express $\frac{dy}{dx}$ in terms of x.

22. Given $y^2=4ax$, to express $\frac{dy}{dx}$ in terms of x, and also in terms of y.

23. One end of a ball of thread is fastened to the top of a pole 35 feet high; a man carrying the ball moves uniformly from the bottom at the rate of five miles an hour, allowing the thread to unwind as he advances; the height of the ball being 5 feet, what is the man's distance from the pole when the thread is unwinding at the rate of one mile per hour?

24. A ladder 25 feet long reclines against a wall, the bottom of the ladder being 7 feet distant from the bottom of the wall; when in this position, a man begins to pull the lower extremity along the ground at the rate of 2 feet per second; at what rate does the other extremity *begin* to descend along the face of the wall?

25. Two engines are moving along two straight lines of railway which intersect at an angle of 60°; the one is approaching the intersection at the rate of 25 miles an hour, and the other is receding from it at the rate of 30 miles an hour; find the rate at which they are separating from each other, if each is 10 miles from the intersection.

26. Derive, by the method employed in Art. 32 to determine the differential of the square, the result $d\left(\frac{1}{x}\right) = \frac{c}{x^2}$, c being an unknown constant.

27. A street-crossing is 10 feet from a street-lamp situated directly above the curbstone, which is 60 feet from the vertical walls of the opposite buildings. If a man is walking across to the opposite side of the street at the rate of 4 miles an hour, at what rate does his shadow move upon the walls—1°. When he is 5 feet from the curbstone; 2°. When he is 20 feet from the curbstone?

Ans. 1°.—96 miles per hour. 2°.—6 miles per hour.

VIII.

LOGARITHMIC DIFFERENTIALS.

38. Formula (*c*) for the differential of a product may be written—

$$\frac{d(xy)}{xy}=\frac{dx}{x}+\frac{dy}{y}\dots\dots\dots\dots\dots\dots(1)$$

This form is readily extended to products of any number of factors, thus—

let $x_1\,x_2\,x_3\dots\dots x_p$ denote the product of p variable factors, then—

$$\frac{d(x_1\,x_2\,x_3\dots\dots x_p)}{x_1\,x_2\,x_3\dots\dots x_p}=\frac{dx_1}{x_1}+\frac{d(x_2\,x_3\dots\dots x_p)}{x_2\,x_3\dots\dots x_p},$$

$$=\frac{dx_1}{x_1}+\frac{dx_2}{x_2}+\frac{d(x_3\dots\dots x_p)}{x_3\dots\dots x_p},$$

finally, $$\frac{d(x_1\,x_2\,x_3\dots\dots x_p)}{x_1\,x_2\,x_3\dots\dots x_p}=\frac{dx_1}{x_1}+\frac{dx_2}{x_2}+\frac{dx_3}{x_3}\dots\dots\frac{dx_p}{x_p}\dots\dots(2)$$

From the analogy of the property thus expressed to the fundamental property of logarithms, *the ratio of its differential to the variable* is called the *logarithmic differential*; thus equation (2) expresses that the *logarithmic differential of the product of any number of variable factors is equal to the sum of the logarithmic differentials of the factors.*

Equation (2) cleared of fractions becomes

$$d(x_1\,x_2\dots x_p)=x_2\,x_3\dots x_p\,dx_1+x_1\,x_3\dots x_p\,dx_2+x_1\,x_2\dots dx_p,\dots(c')$$

an extension of formula (*c*).

39. *The logarithmic differential of the reciprocal* is found thus—

$$\frac{d\left(\frac{1}{x}\right)}{\frac{1}{x}}=\frac{-\frac{dx}{x^2}}{\frac{1}{x}}=-\frac{dx}{x}\dots\dots\dots\dots\dots\dots(3)$$

That is, *the logarithmic differential of the reciprocal of a variable is simply the negative of the logarithmic differential of the variable.* This is likewise analogous to a property of logarithms.

Equation (3) enables us to extend the general property of logarithmic differentials to the case of factors which appear in the denominator, the logarithmic differentials of such factors being subtractive. Hence we may write

$$d\left(\frac{x_1\, x_2 \ldots x_p}{y_1\, y_2 \ldots y_p}\right)=\frac{dx_1}{x_1}+\frac{dx_2}{x_2}\ldots+\frac{dx_p}{x_p}-\frac{dy_1}{y_1}-\frac{dy_2}{y_2}\ldots-\frac{dy_p}{y_p}\ldots(4)$$

40. *The logarithmic differential of the Power* is obtained in the following manner:

First, in equation (2) suppose the variables $x_1\, x_2\, x_3 \ldots x_p$ to be each replaced by x; the first number contains p factors, and the second p terms, the equation therefore reduces to

$$\frac{d(x^p)}{x^p}=p\frac{dx}{x}\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Again, the logarithmic differential of a root of a variable is found by dividing the logarithmic differential of the variable by the index of the root; for (x being an integral power of its root) we have by equation (5),

$$\therefore \frac{dx}{x}=p\frac{d(\sqrt[p]{x})}{\sqrt[p]{x}},$$

hence,
$$\frac{d(\sqrt[p]{x})}{\sqrt[p]{x}}=\frac{1}{p}\cdot\frac{dx}{x};$$

or, adopting the notation of fractional exponents,

$$\frac{d(x^{\frac{1}{p}})}{x^{\frac{1}{p}}}=\frac{1}{p}\cdot\frac{dx}{x}\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

In general, the exponent being any fraction $\frac{p}{q}$, we have, from (5) and (6),

$$\frac{d(x^{\frac{p}{q}})}{x^{\frac{p}{q}}} = \frac{1}{q}\cdot\frac{d(x^p)}{x^p} = \frac{p}{q}\cdot\frac{dx}{x} \dots\dots\dots\dots\dots (7)$$

Again, by applying equation (3), since $x^{-n} = \frac{1}{x^n}$, we obtain the logarithmic differential of a power when the exponent is negative, thus—

$$\frac{d(x^{-n})}{x^{-n}} = -\frac{d(x^n)}{x^n} = -n\frac{dx}{x} \dots\dots\dots\dots (8)$$

Hence n denoting any number, integral or fractional, positive or negative, we have by (5), (6), (7), and (8),

$$\frac{d(x^n)}{x^n} = n\frac{dx}{x} \dots\dots\dots\dots\dots (9)$$

That is, *the logarithmic differential of the* n^th^ *power of a variable is equal to* n *times the logarithmic differential of the variable.*

41. The differentials of many expressions are most readily obtained by writing first the value of the logarithmic differential; thus, if

$$u = \frac{x^3(a-x)\sqrt{y+b}}{\sqrt[3]{y}(x+a)^{\frac{3}{2}}},$$

$$\frac{du}{u} = 3\frac{dx}{x} - \frac{dx}{a-x} + \tfrac{1}{2}\frac{dy}{y+b} - \tfrac{1}{3}\frac{dy}{y} - \tfrac{3}{2}\frac{dx}{x+a}$$

from which, by substituting the value of u, du may be obtained. This method is adapted to the differentiation of those expressions involving variable factors, to which logarithms are adapted in computation.

THE POWER (x^n).

42. The formula for differentiating any power can now be obtained by multiplying equation (9) Art. 40 by x^n, thus—

$$d(x^n) = nx^{n-1}\,dx \,\ldots\ldots\ldots\ldots\ldots\ldots (f)$$

That is, *the differential of a power of any variable is the continued product of the exponent, the variable with an exponent less by one than the given exponent, and the differential of the variable.*

Thus--

$$d\left(\frac{1}{x^6}\right) = d(x^{-6}) = -\,6x^{-7}\,dx,$$

and
$$d\left(x^{\frac{2}{3}}\right) = \tfrac{2}{3}\,x^{-\frac{1}{3}}\,dx.$$

This result applies of course to powers of functions of one or more variables. Thus—

$$d(ay + bx^n)^m = m(ay + bx^n)^{m-1}(ady + bnx^{n-1}dx).$$

Examples VIII.

1. Differentiate $3ax^5+b^4$, and $\sqrt{x^3 - a^3}$.

2. Differentiate $(a + bx)x^3$, and $(1 + 2x^2)(1 + 4\,x)^3$. [Expand before differentiating.]

3. Differentiate $(1 + x)^4\;(1 + x^2)^2$.

4. Differentiate the identities $x^m\,x^n = x^{m+n}$, and $(x^m)^{\frac{1}{n}} = x^{\frac{m}{n}}$.

5. Differentiate $(a + x)(b + x)(c + x)$. [Expand.]

6. Find the derivative of $(1 + x^m)^n + (1 + x^n)^m$.

7. Find the derivative of $\dfrac{x^3}{x + \sqrt{1 + x^2}}$. [Rationalize the denominator.]

8. Find the derivative of $\frac{x^3}{\sqrt{1-x^2}}$.

9. Find the derivative of $u = (1 + x)\sqrt{1 - x}$.

$$\frac{du}{dx} = \sqrt{1-x} - \frac{1+x}{2\sqrt{1-x}} = \frac{1-3x}{2\sqrt{2-x}}.$$

10. Differentiate $\frac{x^3}{\sqrt{(1-x^2)^3}}$. *Ans.* $\frac{3x^2dx}{(1-x^2)^{\frac{5}{2}}}$.

11. From the equation $xy = 1$, derive the logarithmic differential of y.

12. Derive the logarithmic differential of the quotient $\left(\frac{x}{y}\right)$ directly from formula (e).

13. If $y = \frac{x^n}{(1+x)^n}$, find $\frac{dy}{dx}$.

$$y = \left(\frac{x}{1+x}\right)^n \therefore \frac{dy}{dx} = n\left(\frac{x}{1+x}\right)^{n-1}\frac{1+x-x}{(1+x)^2} = \frac{nx^{n-1}}{(1+x)^{n+1}}.$$

14. If $y = (a+x)^3(b-x)^4x^2$, find dy by formula (c').

15. If $y = (a+x)^m(b+x)^n$, find $\frac{dy}{dx}$.

16. Differentiate $\frac{1}{(a+x)^m} \cdot \frac{1}{(b+x)^n}$. [Use negative exponents.]

17. If $u = \frac{x^n+1}{x^n-1}$, find $\frac{du}{dx}$.

18. If $u = (1-x)^{\frac{1}{2}}(1+x)$, find du.

19. If $u = (3b + 2ax)^{\frac{3}{2}}(b - ax)$, find du.

Ans. $5a^2x\sqrt{3b+2ax}.dx$.

20. If $y = \sqrt{\frac{1+x}{1-x}}$, show that $\frac{dy}{dx} = \frac{1}{(1-x)\sqrt{1-x^2}}$.

21. If $y=\sqrt{\frac{1-x^2}{(1+x^2)^3}}$, find $\frac{dy}{dx}$. [Use logarithmic differentials.] *Ans.* $\frac{-2x(2-x^2)}{(1-x^4)^{\frac{1}{2}}(1+x^2)^2}$.

22. If $u=\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}^n$, prove that $\frac{du}{dx}=\frac{nu}{x\sqrt{1-x^2}}$.

23. If $y=\frac{x}{\sqrt{1-x^2}}\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}^n$, find $\frac{dy}{dx}$.

[Put $\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}^n=u$, and use logarithmic differentials; see example 22.]

$$Ans.\ \frac{dy}{dx}=\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}^n\frac{1+n\sqrt{1-x^2}}{(1-x^2)^{\frac{3}{2}}}$$

24. Differentiate $\frac{bx}{\sqrt{2ax-x^2}}$, in the form $b\left\{\frac{2a}{x}-1\right\}^{-\frac{1}{2}}$. See Ex. VII. 14.

25. Differentiate $\frac{2x^2-1}{x\sqrt{1+x^2}}$.

26. Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{x}$.

[Transform thus, $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{x}=\sqrt{x^{-2}+1}+\sqrt{x^{-2}-1}$.]

$$Ans.\ -\frac{\sqrt{1-x^2}+\sqrt{1+x^2}}{x^2\sqrt{1-x^4}}dx.$$

27. If $u=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, find $\frac{du}{dx}$. [Rationalize the denominator before differentiating.]

$$Ans.\ -\frac{2}{x^3}\left\{1+\frac{1}{\sqrt{1-x^4}}\right\}.$$

28. A tree whose diameter is 6 inches is increasing in height and diameter proportionately; compare the present rate of its volume with the rate of its volume when its diameter, supposed to increase uniformly, shall have become 3 feet.

29. If an ingot of silver in the form of a parallelopiped expand $\frac{1}{1000}$ part of each of its linear dimensions for each degree of temperature, at what rate is its volume increasing when the sides are respectively 2, 3, and 6 inches? [Observe that, x representing a side, dx may be used to denote *the rate per degree of temperature instead of the rate per second.*]

30. Prove generally that, if the coefficient of expansion of each linear dimension of a solid is k, its coefficient of expansion in volume is $3k$.

Let x denote any side; then, if V denote the volume, we shall have $V = cx^3$; c being a constant dependent on the shape of the body,

$$dV = 3cx^2 dx;$$

but $$dx = kx, \therefore dV = 3k.cx^3 = 3k.V.$$

31. Wine is poured into a conical glass 3 inches in height at a uniform rate, filling the glass in 8 seconds; at what rate is the surface rising at the end of one second? At what rate when the surface reaches the brim?

Let h denote the height of the glass (3 inches), v_1 its entire volume, v the volume when filled to the height x, and a the time required to fill the glass (8 seconds).

$$\frac{x^3}{h^3} = \frac{v}{v_1}. \therefore x = h\frac{v^{\frac{1}{3}}}{v_1^{\frac{1}{3}}}, \text{ whence } dx = \frac{h}{3}.\frac{v^{-\frac{2}{3}}}{v_1^{\frac{1}{3}}}dv;$$

but $$dv = \frac{v_1}{a}. \therefore dx = \frac{h}{3a}\left(\frac{v_1}{v}\right)^{\frac{2}{3}}.$$

In this case $$dx = \tfrac{1}{8}\left(\frac{v_1}{v}\right)^{\frac{2}{3}}.$$

If $\frac{v_1}{v} = 8$, $dx = \frac{1}{2}$ inch per second. 1st case.

If $\frac{v_1}{v} = 1$, $dx = \frac{1}{8}$ inch per second. 2d case.

32. A vessel sailing due south at the uniform rate of 8 miles per hour is 20 miles north of a vessel sailing due east at the rate of 10 miles an hour. At what rate are they separating—1°. At the end of $1\frac{1}{2}$ hours? 2°. At the end of $2\frac{1}{2}$ hours?

Ans. 1°. Separating at the rate of $\frac{43}{68}$ of a mile per hour.

33. When are the two ships mentioned in the preceding example neither receding from nor approaching each other? [Put the expression for their rate of separation equal to zero.]

Ans. When the first ship has sailed $7\frac{33}{41}$ miles.

34. If the difference of the n^{th} powers of two variables is constant, show that the $(1 - n)^{\text{th}}$ power of the ratio of the variables is equal to the ratio of their rates.

CHAPTER III.

THE DIFFERENTIATION OF TRANSCENDENTAL FUNCTIONS.

IX.

THE LOGARITHMIC FUNCTION ($\log x$).

43. In this chapter, the formulas for the differentiation of the simple transcendental functions of an independent variable are to be deduced; by means of these, in connection with the formulas already established for algebraic functions, the student will be enabled to differentiate any function expressed by a combination of known symbols.

To deduce the differential of the logarithmic function, we employ the method exemplified in Art. 32.

The symbol $\log x$ is used to denote the logarithm of x to any base, and $\log_b x$ is used when we wish to designate a particular base b.

Let $z = mx$, then will $dz = m dx \ldots\ldots (1)$
and $\log z = \log m + \log x$, $\therefore\ d(\log z) = d(\log x) \ldots\ldots (2)$

Dividing (2) by (1),

$$\frac{d(\log z)}{dz} = \frac{d(\log x)}{m dx};$$

substituting for m its value $\frac{z}{x}$, and separating the variables, we obtain

$$z\,\frac{d(\log z)}{dz} = x\,\frac{d(\log x)}{dx} \ldots\ldots\ldots\ldots (3)$$

This equation, like equation (5) of Art. 32, is true for all values of x, z, dx, and dz. For, the derivatives being functions respectively of z and of x by Art. 27, equation (3) may be written in the form

$$zf'(z) = xf'(x); \qquad \dots\dots(3')$$

and, moreover, the values of x and z are entirely independent, because their assumed ratio m has been eliminated. Now, since the form of the first member of equation (3) or (3′) is the same as that of the second member, this equation shows that the value of the expression $x\dfrac{d(\log x)}{dx}$ is unchanged when x is changed to z, the latter denoting *any other* value of the independent variable, that is, the expression is independent of the value of x; it will be found, however, to depend upon the base of the system of logarithms to which $\log x$ belongs.

Denoting now the base by b, we put

$$x\frac{d(\log_b x)}{dx} = B. \qquad \dots\dots(4)$$

44. In this equation, B can be dependent upon no quantity except b. Equation (4) may be written in the form

$$d(\log_b x) = \frac{B\,dx}{x}; \qquad \dots\dots(5)$$

similarly, if a is the base, we use A to denote the constant value of the expressions in equation (3), thus—

$$d(\log_a x) = \frac{A\,dx}{x}. \qquad \dots\dots(6)$$

A relation between A and B is found by differentiating by (5) and (6), the identical equation

$$\log_a x = \log_a b \, \log_b x,^* \qquad \dots\dots(7)$$

---

* This identity is most readily obtained thus,—by definition

$$x = b^{\log_b x}$$

taking the logarithm to the base a of each member, we have

$$\log_a x = \log_b x \, \log_a b.$$

and thus obtaining, $\dfrac{A\,dx}{x} = \log_a b\,\dfrac{B\,dx}{x}$;

whence $A = B\log_a b = \log_a b^B$;

therefore, by the definition of a logarithm,

$$a^A = b^B \dots\dots\dots\dots\dots\dots (8)$$

Now, it is obvious that the value of a^A cannot depend upon b, hence equation (8) shows that the value of b^B likewise cannot depend upon b; b^B must, therefore, have a value entirely independent of the base of the system of logarithms to which $\log x$ belongs. Denoting this constant value by e, we write

$$b^B = e \dots\dots\dots\dots\dots\dots (9)$$

Taking the logarithms of each member of equation (9) adopting this constant as a base, we have

$$B \log_e b = 1,$$

whence $$B = \frac{1}{\log_e b} \dots\dots\dots\dots\dots\dots (10)$$

Introducing this value of B in equation (5), we obtain

$$d(\log_b x) = \frac{dx}{\log_e b \,.\, x} \dots\dots\dots\dots\dots\dots (g)$$

In this equation, the differential of a logarithm to any base is expressed with the aid of the unknown constant e.

If the logarithms are taken in the system whose base is e, equation (g) reduces to

$$d(\log_e x) = \frac{dx}{x} \dots\dots\dots\dots\dots\dots (g')$$

NEPERIAN OR HYPERBOLIC LOGARITHMS.

45. The constant e is known as the Neperian base [from the Latin form of the name of NAPIER, the inventor of logarithms]; logarithms to this base are variously termed natural, Neperian, or hyperbolic logarithms.

On account of the simple form of equation (g') these logarithms are usually employed in deducing formulas; but, as logarithms to the base 10 are exclusively used in computation, the resulting formulas are rendered available for practical purposes, by substituting for $\log_e x$ its value from the equation

$$\log_{10} x = \log_{10} e \log_e x,$$

which is deduced from equation (7), Art. 44, by writing 10 for a, and e for b.

Hereafter whenever the base of logarithms is not expressed, e is to be understood.

The term *hyperbolic* as applied to Neperian logarithms is derived from a property of the rectangular hyperbola deduced in articles 46 and 47.

46. The equation of the hyperbola referred to its asymptotes is

$$xy = c^2;$$

taking $c = 1$, and the axes rectangular, we have for the equation of the rectangular hyperbola in the figure,

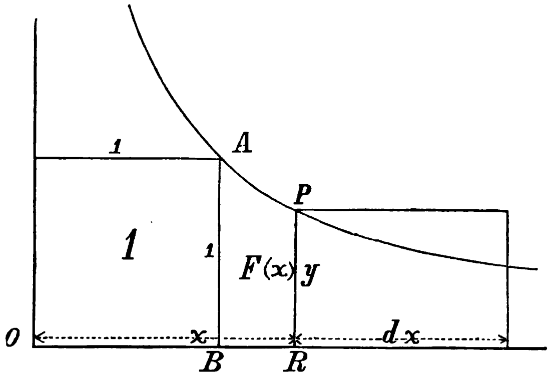
$$xy = 1 \text{ or } y = \frac{1}{x}.$$

Draw the ordinate to the vertex (1, 1), and suppose a variable ordinate to move from this position toward the right. The area generated by this ordinate will be an increasing variable quantity, which may be regarded as a function of x and denoted by $F(x)$.

At the instant x has attained a given value, the corresponding rate of $F(x)$ may be measured by giving the area a continuously uniform rate identical with that which it has at that instant [see Art. 8]. This is effected by supposing y to remain constant and x to increase uniformly from and after the given instant with the rate dx, thus rendering the rate of the area uniform. In the figure the area $ABRP$ represents the value of $F(x)$ at the given instant, and the rectangle ydx its *rate* at the same instant. Hence we have at every instant

$$d\{F(x)\} = d(\text{Area}) = ydx.$$

This equation, it may be remarked, has been derived without regard to the relation between y and x, and is therefore applicable, whatever be the equation of the curve.



47. In the present case, since $y = \frac{1}{x}$, we have

$$d\{F(x)\} = \frac{dx}{x} = d(\log_e x),$$

by equation (g'). Thus, $F(x)$, the expression for the area, varies at the same rate as the Neperian logarithm of x.

Now, when $x = 1$, the area reduces to zero, and $\log_e 1 = 0$,

or $$F(1) = \log_e 1\,;$$

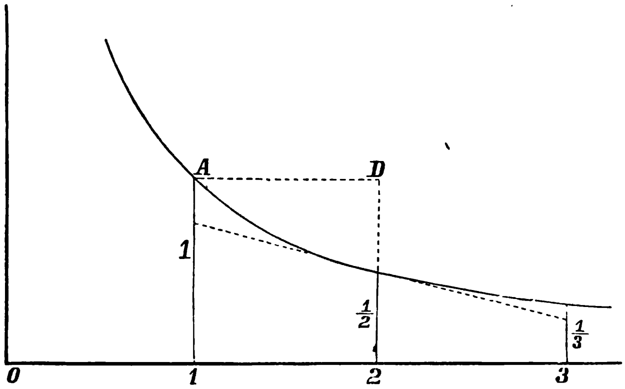
therefore, $F(x)$ and $\log_e x$ starting from the same value, and constantly increasing at the same rate, we have

$$F(x) = \log_e x.$$

That is, the area included betweent he hyperbola, the axis of x, and the ordinates corresponding to x and 1 respectively is equal to the Neperian logarithm of x. It is easily inferred by subtraction that *the area included between any two ordinates is the logarithm of the ratio of the abscissas.*

48. Let the ordinates corresponding to the abscissas 2 and 3 be drawn; then the area included between the ordinates corresponding to 1 and 2 will represent $\log_e 2$, and that included between those corresponding to 1 and 3 will represent $\log_e 3$. Completing the square $A\,D\,2\,1$, we see that $\log_e 2$ is less than one.

Draw a tangent to the curve at the point $(2, \frac{1}{2})$; this tangent, the axis of x, and the ordinates corresponding to 1 and 3 form a trapezoid of which the sum of the parallel sides is one, and the distance between them two; its area is, therefore, one. The area representing $\log_e 3$ being greater than this trapezoid we have $\log_e 3$ greater than one;



therefore $\log_e 3 > 1 > \log_e 2.$

Now, since $\log_e e = 1$, this inequality is equivalent to

$$\log_e 3 > \log_e e > \log_e 2$$

or $$3 > e > 2.$$

The Neperian base is therefore intermediate in value between 2 and 3; its *exact value* will be derived in a succeeding chapter.

49. Formula (g') shows that the differential of the Neperian logarithm of a variable is identical with the expression defined in Art. 38 as the logarithmic differential of the variable.

Negative quantities have no real logarithms, but since

$$\frac{d(-x)}{-x} = \frac{dx}{x},$$

the logarithmic differential of a negative quantity is the differential of the Neperian logarithm of its numerical value taken positively.

THE EXPONENTIAL FUNCTION (a^x).

50. An expression of the form a^x, in which x denotes any variable, and a any constant, is called an exponential function.

If a is negative, a^x cannot denote a continuously varying quantity. We therefore exclude the case in which a has negative values, and regard a^x as a continuously varying positive quantity, of which the logarithm to the base a is x.

Assume $$y = a^x,$$

whence $$\log_a y = x;$$

differentiating by (g), $$\frac{dy}{\log_e a . y} = dx,$$

or $$dy = \log_e a . y dx;$$

but $$y = a^x \therefore d(a^x) = \log_e a . a^x dx \dots\dots\dots\dots (h).$$

If $a = e$ we have as a particular case

$$d(e^x) = e^x dx \dots\dots\dots\dots\dots\dots (h').$$

51. An expression of the form x^y, in which x and y denote any two variables, may easily be transformed into an ordinary exponential function: thus, by definition of log x,

$$x = e^{\log x},$$

$$\therefore \quad x^y = e^{y \log x}.$$

In this form the expression can be differentiated by formula (h'), thus,—

$$d(e^{y \log x}) = e^{y \log x} \, d(y \log x),$$

or

$$d(x^y) = x^y (y \frac{dx}{x} + \log x . dy).$$

We may also differentiate x^y by assuming

$$z = x^y,$$

whence, taking Neperian logarithms,

$$\log z = y \log x.$$

By formula (g')

$$\frac{dz}{z} = y \frac{dx}{x} + \log x . dy,$$

$$dz = z \, [y \frac{dx}{x} + \log x . dy],$$

or

$$d(x^y) = x^y \, [y \frac{dx}{x} + \log x . dy]$$

as before.

Examples IX.

1. For what value of $\log_a x$ are the differentials of the function and of the variable equal?

2. Trace the curve $y = a^x$, taking $a = 2$; also taking $a = 3$.

3. What is the value of $\tan \varphi$ at the point in which these curves cut the axis of y? What is its value at the same point, if $y = e^x$?

4. Trace the curve $y = \log_a x$, and prove that if the point whose abscissa is e be joined to the origin, the joining line will be a tangent to the curve.

5. Prove formula (c') by differentiating the logarithms of the two members of the equation $u = x_1 x_2 \dots x_p$.

6. Prove formula (e) in a similar manner; also formulas (d) and (f).

7. Differentiate the identity $a^{x+y} = a^x a^y$.

8. Differentiate the identity $(a^m)^x = (a^x)^m = a^{mx}$.

9. What inference in regard to the differentiation of logarithmic expressions can be drawn from the differentiation of the identity

$$\log\frac{bx}{a} = \log b - \log a + \log x\,?$$

10. Differentiate a^{x^2}. *Ans.* $2 \log a . a^{x^2} x\, dx$.

11. Differentiate $e^{\frac{1}{1+x}}$.

12. Find the differential of a^{b^x}.

Ans. $\log a . \log b . a^{b^x} . b^x . dx$.

13. $x \log x$. *Ans.* $dx + \log x . dx$.

14. $e^x(1 - x^3)$. *Ans.* $e^x(1 - 3x^2 - x^3)\, dx$.

15. $(x - 3)\, e^{2x} + 4xe^x + x + 3$.

Ans. $[(2x - 5)\, e^{2x} + 4\,(x + 1)\, e^x + 1]\, dx$.

16. $(2x - 5)\, e^{2x} + 4\,(x + 1)\, e^x - 1$.

Ans. $4e^x\{(x - 2)\, e^x + x + 2\}\, dx$.

17. $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. *Ans.* $\dfrac{4\, dx}{(e^x + e^{-x})^2}$.

18. $\log (x + \sqrt{x^2 + a^2})$. *Ans.* $\dfrac{dx}{\sqrt{x^2 + a^2}}$.

19. $\log (e^x + e^{-x})$.

20. $\log (\log x)$. *Ans.* $\dfrac{dx}{x \log x}$.

21. $\log \{\log (a + bx^n)\}$.

22. $a^{\log x}$.

23. $\log \dfrac{x}{x - \sqrt{1 - x^2}}$.

Ans. $\left\{\dfrac{1}{x} - \dfrac{x + \sqrt{1 - x^2}}{[x - \sqrt{1 - x^2}]\sqrt{1 - x^2}}\right\} dx$.

24. $\log \dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}}$ *Ans.* $\dfrac{\sqrt{a}dx}{(a - x)\sqrt{x}}$.

25. $\log \dfrac{e^x}{e^x + 1}$.

26. Prove that $\log_{a^n} x^n = \log_a x$.

27. If $f'(x) = \dfrac{1}{x - a}$, prove that $f(x) = \log c\,(a - x)$ or $\log c\,(x - a)$, according as $a > x$ or $a < x$, c denoting any positive constant.

28. $\sqrt{x} - \log (\sqrt{x} + 1)$. *Ans.* $\dfrac{dx}{2(\sqrt{x} + 1)}$.

29. x^x.

30. $x^{\log x}$.

X.

THE TRIGONOMETRIC OR CIRCULAR FUNCTIONS.

52. In expressing the differentials of the circular functions of a variable angle, we employ, as the measure of the angle, the ratio of the subtending arc to its radius; in other words, the quotient obtained by dividing the length of the arc expressed in linear units by the length of the radius. Let s denote the length of the arc, a that of the radius, and let θ be the circular measure of the angle; then

$$\theta = \frac{s}{a}.$$

The differentials of the circular functions of θ, viz.:

$$\begin{array}{lll} \sin\theta & \tan\theta & \sec\theta \\ \cos\theta & \cot\theta & \operatorname{cosec}\theta \end{array}$$

are now to be expressed in terms of $d\theta$.

When the independent variable θ reaches the value 2π, the values of its functions return to the values with which they started when θ was zero; hence these functions are said to be *periodic*; their derivatives likewise partake of the same character.

THE SINE AND THE COSINE ($\sin\theta$ and $\cos\theta$).

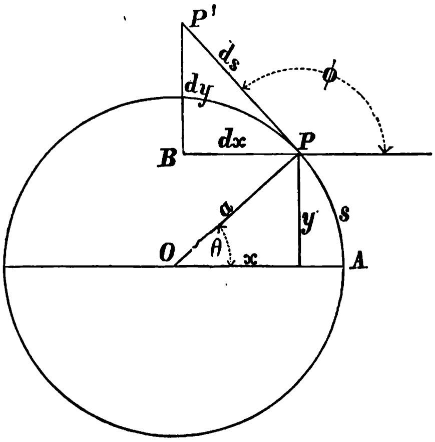
53. Let the variable angle θ be generated by the rotation of the radius a about the origin of rectanglar co-ordinates starting from the position OA. The extremity P of the radius moves in the circle

$$x^2 + y^2 = a^2, \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and generates the variable arc s.

Let PB and BP' represent the rates respectively of the abscissa and ordinate of P when it arrives at the position indicated in the figure. Then will PP′ be a tangent as in Art. 25;

and, moreover, the length of PP′ will represent the actual velocity of the point P, or ds, because it is the space through which P would move in the unit of time, were dx, dy, and consequently ds to become constant.



Now, $$\sin\theta = \frac{y}{a}, \quad \text{and} \quad \cos\theta = \frac{x}{a}, \ldots\ldots\ldots\ldots (2)$$

$$\therefore \quad d(\sin\theta) = \frac{dy}{a}, \text{ and } d(\cos\theta) = \frac{dx}{a} \ldots\ldots\ldots\ldots (3)$$

In equations (3) we have to express dy and dx in terms of θ and $d\theta$.

Denoting by φ (see Art. 26) the inclination to the axis of x of the direction of the motion of the generating point when s is *increasing*, we have

$$dy = \sin\varphi.\ ds, \quad \text{and} \quad dx = \cos\varphi.\ ds, \ldots\ldots\ldots\ldots (4)$$

and, substituting in equations (3), we obtain

$$d(\sin\theta) = \sin\varphi\,\frac{ds}{a}, \quad \text{and} \quad d(\cos\theta) = \cos\varphi\,\frac{ds}{a} \ldots\ldots\ldots (5)$$

In the figure, φ being in the second quadrant, and ds being positive, dx is negative.

Differentiating equation (1)

$$xdx + ydy = 0,$$

whence $$\tan\varphi = \frac{dy}{dx} = -\frac{x}{y}; \qquad (6)$$

or, since $$\frac{y}{x} = \tan\theta, \qquad (7)$$

$$\tan\varphi = -\cot\theta = \tan(\theta \pm \frac{\pi}{2});$$

therefore $$\varphi = \theta + \frac{\pi}{2}. \qquad (8)$$

In equation (8) we take $\theta + \frac{\pi}{2}$, because φ has been defined as the angle between the positive directions of ds and dx.

Now, in equations (5), we substitute

$$\sin\varphi = \sin(\theta + \frac{\pi}{2}) = \cos\theta,$$

$$\cos\varphi = \cos(\theta + \frac{\pi}{2}) = -\sin\theta,$$

and $$\frac{ds}{a} = d\theta, \text{ since } \frac{s}{a} = \theta;$$

whence $$d(\sin\theta) = \cos\theta d\theta, \qquad (i)$$

and $$d(\cos\theta) = -\sin\theta d\theta. \qquad (j)$$

54. Equation (8) shows that in the case of the circle the tangent, according to the general definition of Art. 25, is perpendicular to the radius through the point of contact, and is therefore identical with the tangent as defined in Geometry.

THE TANGENT AND THE COTANGENT ($\tan\theta$ and $\cot\theta$).

55. *The differential* of $\tan\theta$ is found by applying formula (*e*) to the equation

$$\tan\theta = \frac{\sin\theta}{\cos\theta};$$

thus $$d(\tan\theta) = \frac{\cos\theta\, d(\sin\theta) - \sin\theta\, d(\cos\theta)}{\cos^2\theta},$$

or $$d(\tan\theta) = \frac{d\theta}{\cos^2\theta} = \sec^2\theta\, d\theta \ldots\ldots\ldots\ldots (k)$$

The differential of $\cot\theta$ is found by applying formula (*k*) to the equation

$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right);$$

whence $$d(\cot\theta) = -\frac{d\theta}{\sin^2\theta} = -\operatorname{cosec}^2\theta\, d\theta \ldots\ldots\ldots\ldots (l)$$

THE SECANT AND THE COSECANT ($\sec\theta$ and $\operatorname{cosec}\theta$).

56. *The differential* of $\sec\theta$ is found by applying formula (*d*) to the equation

$$\sec\theta = \frac{1}{\cos\theta};$$

whence $$d(\sec\theta) = \frac{\sin\theta\, d\theta}{\cos^2\theta} = \sec\theta\, \tan\theta\, d\theta \ldots\ldots\ldots\ldots (m)$$

The differential of $\operatorname{cosec}\theta$ is found by applying formula (*m*) to the equation

$$\operatorname{cosec}\theta = \sec\left(\frac{\pi}{2} - \theta\right);$$

whence $$d(\operatorname{cosec}\theta) = -\frac{\cos\theta\, d\theta}{\sin^2\theta} = -\operatorname{cosec}\theta\, \cot\theta \ldots\ldots\ldots (n)$$

EXAMPLES X.

1. By means of the equation $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$, derive the formula for differentiating $\cos\theta$.

2. By means of the equation $\text{versin}\,\theta = 1 - \cos\theta$, derive a formula for differentiating $\text{vers}\,\theta$.

3. Why have the derivatives of the complemental functions the negative sign.

4. Show that the function $\tan\theta$ is always increasing as θ increases. Show also how this is consistent with the periodic character of $\tan\theta$.

5. For what value of θ is $d(\sin\theta) = d(\tan\theta)$?

Ans. $\theta = 0$.

6. From the identity $\sin^2\theta + \cos^2\theta = 1$, derive the differential of $\cos\theta$.

7. From the identity $\sec^2\theta = 1 + \tan^2\theta$, derive the differential of $\sec\theta$.

8. What trigonometric formula is obtained by taking the derivatives of the two members of the equation

$$\sin 2\theta = 2\sin\theta\cos\theta\,?$$

9. From the identity $\sin\left(\theta + \frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2}\,(\sin\theta + \cos\theta)$, dederive another by taking derivatives.

10. Find the differentials of

$\log\sin\theta \qquad \log\tan\theta \qquad \log\sec\theta$

$\log\cos\theta \qquad \log\cot\theta \qquad \log\text{cosec}\,\theta.$

11. $l^{\sin x}$. *Ans.* $l^{\sin x}\cos x\,dx$.

12. $\sin e^x$.

13. $x\sin x^2$. *Ans.* $(\sin x^2 + 2x^2\cos x^2)dx$.

14. $\theta + \sin\theta \cos\theta$. *Ans.* $2\cos^2\theta \, d\theta$.

15. $\sin\theta - \frac{1}{3}\sin^3\theta$.

16. $\tan^2\theta + \log(\cos^2\theta)$. *Ans.* $2\tan^3\theta \, d\theta$.

17. $\log(\sec\theta + \tan\theta)$.

18. $\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$.

19. $\frac{1}{3}\tan^3\theta - \tan\theta + \theta$.

20. $\theta + \log\cos\left(\frac{\pi}{4} - \theta\right)$.

21. $\tan a^{\frac{1}{x}}$.

22. $\sin n\theta \, (\sin\theta)^n$. *Ans.* $n(\sin\theta)^{n-1}\sin(n+1)\theta \, d\theta$.

23. $\log\sqrt{\sin x} + \log\sqrt{\cos x}$. *Ans.* $\cot 2x \, dx$.

24. $e^{ax}(a\sin x - \cos x)$. *Ans.* $(a^2 + 1)\, e^{ax}\sin x \, dx$.

25. $e^x(\cos x - \sin x)$. *Ans.* $-2e^x \sin x \, dx$.

26. The crank of a small steam-engine is one foot in length, and revolves uniformly at the rate of two turns per second; the connecting-rod being 5 feet in length; find the velocity of the piston when the crank makes an angle of 45° with the line of motion of the piston-rod; also when the angle is 135°, and when it is 90°.

Let a, b, and x denote respectively the crank, connecting-rod, and variable side of the triangle; and let θ denote the angle between a and x.

We easily deduce

$$x = a\cos\theta + \sqrt{b^2 - a^2\sin^2\theta};$$

whence

$$dx = -\left(a\sin\theta + \frac{a^2\sin\theta\cos\theta}{\sqrt{b^2 - a^2\sin^2\theta}}\right)d\theta.$$

In this case, $d\theta = 4\pi$, $a = 1$, and $b = 5$.

27. Obtain an identity by taking derivatives of both members of the equation

$$\tan\tfrac{1}{2}\theta = \frac{1-\cos\theta}{\sin\theta},$$

and verify the result.

28. An elliptical cam revolves at the rate of two turns per second about a perpendicular axis passing through one of the foci, and gives a reciprocating motion to the end of a bar moving in vertical guides in a line with the centre of rotation. The relation between a radius vector from the focus and its inclination to the major axis is expressed by the equation

$$r = \frac{p}{1-e\cos\theta}.$$

If p is 6 inches and e is $\frac{1}{2}$, find the velocity with which the bar is moving when θ is 60°; also when θ is 90°.

29. Find a general expression for the rate per minute of angle, of $\log_{10}\tan\theta$, and compare the value of this rate when θ is 45° with its value when θ is 15°.

[In this case $d\theta = \dfrac{\pi}{180\times 60}$.]

30. Given, $x = r\cos\theta$, and $y = r\sin\theta$, to deduce

$$dy\,\sin\theta + dx\,\cos\theta = dr,$$

and

$$dy\,\cos\theta - dx\,\sin\theta = r\,d\theta.$$

31. From $x = r\cos\theta$, and $y = r\sin\theta$, deduce

$$(dx)^2 + (dy)^2 = (dr)^2 + r^2(d\theta)^2.$$

32. Trace the curves, $y = \sin x$, $y = \tan x$, and $y = \sec x$.

XI.

INVERSE FUNCTIONS.

57. If we have $y = f(x), \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$
x may be expressed as a function of y, thus—

$$x = \varphi(y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Each of these functions f and φ is called the *inverse* of the other. For example, if

$$y = \log_a x,$$
$$x = a^y.$$

Each of these two functions (the logarithmic and the exponential) is, therefore, the inverse of the other.

If we can solve the equation

$$y = f(x)$$

for x, we thereby find the form of the inverse function; thus, if

$$y = x^2,$$
$$x = \pm \sqrt{y}.$$

When, as in this case, the function derived has more than one form, either may be regarded as the inverse of the given function.

The derivatives of (1) and (2) have the forms

$$\frac{dy}{dx} = f'(x), \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

and
$$\frac{dx}{dy} = \varphi'(y); \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

hence, multiplying, $1 = f'(x) . \varphi'(y) \ldots\ldots\ldots\ldots\ldots (5)$

If, then, the derivative of one of these functions be known, the derivative of its inverse may be deduced from (5).

We have already employed this method in deriving formulas (b) and (h); in each case $\varphi'(y)$ was previously known, and the final step was its expression in terms of x.

INVERSE TRIGONOMETRIC FUNCTIONS.

58. The inverse trigonometric functions require a peculiar notation ; thus,

if $x = \sin\theta, \qquad \theta = \sin^{-1}x,$

and if $x = \tan\theta, \qquad \theta = \tan^{-1}x;$

$\sin^{-1}x$ is read "arc whose sine is x," &c.

While the arc θ increases indefinitely, each of its trigonometric functions passes through all its possible values an indefinite number of times; the number of arcs corresponding to a given value of the function is therefore unlimited. As each function passes through all its possible values once, one value of the corresponding arc passes continuously through a semicircle. Hence, to avoid ambiguity, the symbols $\sin^{-1}x$, $\tan^{-1}x$, $\cos^{-1}x$, &c., are restricted in meaning to the arc within *one such semicircle.*

59. The semicircle is, in each case, so selected as to include the first quadrant, in which therefore the arc is taken, whenever x is positive.

Accordingly, while x passes continuously from -1 through zero to $+1$, $\sin^{-1}x$ passes from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$; this function being impossible for values of x not included between these limits.

In like manner, $\tan^{-1}x$ varies between the same limits, while x passes continuously through all values.

On the other hand, $\sec^{-1}x$ passes from 0 to π, while x passes from $+1$ to -1 through infinity.

60. Since the cosine of the complement of an arc is the same as the sine of the arc, we have the equation

$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x.$$

Likewise, $$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x,$$

and $$\operatorname{cosec}^{-1}x = \frac{\pi}{2} - \sec^{-1}x.$$

The limiting values of these arcs may be derived as in Art. 59, or from the above formulas; the complement of a negative angle being in the second quadrant.

Thus $\sin^{-1}x$, $\tan^{-1}x$, and $\operatorname{cosec}^{-1}x$ vary between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$; while $\cos^{-1}x$, $\cot^{-1}x$, and $\sec^{-1}x$ vary between 0 and π.

It is further evident from these formulas, that the differentials of the last three are simply those of the first three with their signs reversed; it is therefore only necessary to deduce the differentials of $\sin^{-1}x$, $\tan^{-1}x$, and $\sec^{-1}x$.

THE INVERSE SINE ($\sin^{-1}x$).

61. Put $y = \sin^{-1}x$, whence $\sin y = x$.
By differentiating, we obtain

$$dy = \frac{dx}{\cos y}.$$

$\cos y$ is positive, since y is always between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$, therefore $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$, and consequently

$$d(\sin^{-1}x) = \frac{dx}{\sqrt{1 - x^2}} \dots\dots\dots\dots (o)$$

By Art. 60, $$d(\cos^{-1}x) = -\frac{dx}{\sqrt{1 - x^2}} \dots\dots\dots\dots (o')$$

THE INVERSE TANGENT ($\tan^{-1}x$).

62. Put $y = \tan^{-1}x$, whence $\tan y = x$.
By differentiating, we obtain

$$dy = \frac{dx}{\sec^2 y};$$

but $\sec^2 y = 1 + x^2$, therefore

$$d(\tan^{-1}x) = \frac{dx}{1+x^2}.\dots\dots\dots\dots(p)$$

By Art. 60, $$d(\cot^{-1}x) = -\frac{dx}{1+x^2}.\dots\dots\dots\dots(p')$$

THE INVERSE SECANT ($\sec^{-1}x$).

63. Put $y = \sec^{-1}x$, whence $\sec y = x$.
By differentiating, we obtain

$$dy = \frac{dx}{\tan y \sec y}.$$

$\sec y = x$, and $\tan y = \pm\sqrt{x^2 - 1}$; since y is in the first or second quadrant, $\tan y$ has the same sign as $\sec y$, and therefore their product is always positive.

Substituting, we obtain

$$d(\sec^{-1}x) = \frac{dx}{x\sqrt{x^2-1}},\dots\dots\dots\dots(q)$$

and, by Art. 60, $$d(\operatorname{cosec}^{-1}x) = -\frac{dx}{x\sqrt{x^2-1}}.\dots\dots\dots\dots(q')$$

In each of these fractions the denominator $[x\sqrt{x^2-1}]$ must be taken positive even when x is negative.

THE INVERSE VERSED-SINE ($\operatorname{vers}^{-1}x$).

64. Since it is sometimes convenient to use the function $\operatorname{vers}^{-1}x$, we proceed to deduce its differential.

If $x = \operatorname{vers} y$, $y = \operatorname{vers}^{-1}x$,

then $1 - x = \cos y$, $y = \cos^{-1}(1-x)$;

hence $\operatorname{vers}^{-1}x = \cos^{-1}(1-x)$.

Differentiating, $d(\text{vers}^{-1}x) = -\dfrac{d(1-x)}{\sqrt{1-(1-x)^2}}$,

or $$d(\text{vers}^{-1}x) = \frac{dx}{\sqrt{2x-x^2}}. \qquad\qquad (r)$$

EXAMPLES XI.

1. Prove that $d\left(\sin^{-1}\dfrac{mx}{a}\right) = \dfrac{mdx}{\sqrt{a^2-m^2x^2}}$.

2. Prove that $d\left(\tan^{-1}\dfrac{mx}{a}\right) = \dfrac{madx}{a^2+m^2x^2}$.

3. Prove that $d\left(\sec^{-1}\dfrac{mx}{a}\right) = \dfrac{adx}{x\sqrt{m^2x^2-a^2}}$.

4. Derive $d(\cos^{-1}x)$, assuming $y = \cos^{-1}x$.

5. Derive $d(\sec^{-1}x)$ from the equation

$$\sec^{-1}x = \cos^{-1}\frac{1}{x}.$$

6. Show by differentiation that $x = \sin(\pi - \sin^{-1}x)$.

7. Trace the curves, $y = \sin^{-1}x$, $y = \tan^{-1}x$, and $y = \sec^{-1}x$. [Notice the restrictions of Art. 59.]

8. Differentiate $\sin^{-1}2x$, and $\tan^{-1}x^2$.

9. $\sin^{-1}(\cos x)$. *Ans.* $-dx$.

10. $\sin(\cos^{-1}x)$. *Ans.* $-\dfrac{xdx}{\sqrt{1-x^2}}$.

11. $\sin^{-1}(\tan\theta)$. *Ans.* $\dfrac{\sec^2\theta\, d\theta}{1-\tan^2\theta}$.

12. $\cos^{-1}(2\cos\theta)$.

13. Differentiate the identity $\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{x^2+1}}$.

14. Differentiate the identity $\tan^{-1}x+\tan^{-1}a=\tan^{-1}\frac{x+a}{1-ax}$.

15. If $y = \tan^{-1}\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)}$, find $\frac{dy}{dx}$.

16. Find a simpler expression for $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, by examining its differential and its value when $x = 0$.

17. $x\sin^{-1}x + \sqrt{1-x^2}$. *Ans.* $\sin^{-1}x.dx$.

18. $\tan^{-1}e^x$.

19. $(x^2+1)\tan^{-1}x - x$. *Ans.* $2x\tan^{-1}x.dx$.

20. $a^2\sin^{-1}\frac{x}{a} + x\sqrt{a^2-x^2}$. *Ans.* $2\sqrt{a^2-x^2}\,dx$.

21. $\tan x\tan^{-1}x$.

22. $\tan^{-1}\frac{2x}{1-x^2}$. *Ans.* $\frac{2\,dx}{1+x^2}$.

23. $\sin^{-1}\frac{x+1}{\sqrt{2}}$. *Ans.* $\frac{dx}{\sqrt{1-2x-x^2}}$.

24. $\tan^{-1}\frac{x}{\sqrt{1-x^2}}$. *Ans.* $\frac{dx}{\sqrt{1-x^2}}$.

25. $\tan^{-1}(n\tan x)$.

26. $\sec^{-1}\frac{a}{\sqrt{a^2-x^2}}$. *Ans.* $\frac{dx}{\sqrt{a^2-x^2}}$.

27. $(x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax}$. *Ans.* $\tan^{-1}\sqrt{\frac{a}{x}}\,dx$.

28. $\sin^{-1}\sqrt{\sin x}$. *Ans.* $\frac{1}{2}\sqrt{1+\operatorname{cosec} x}\,dx$.

29. $\sqrt{1-x^2}\,\sin^{-1}x - x$. *Ans.* $-\frac{x\sin^{-1}x}{\sqrt{1-x^2}}dx$.

30. $\frac{x\sin^{-1}x}{\sqrt{1-x^2}}+\log\sqrt{1-x^2}$. *Ans.* $\frac{\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx$.

31. Given $x = r\cos\theta$, and $y = r\sin\theta$; eliminate θ and find dr; also eliminate r and find $d\theta$.

32. By means of the values of dr and $d\theta$ required in example 31, prove that

$$(dr)^2 + r^2(d\theta)^2 = (dx)^2 + (dy)^2.$$

33. When light passes from air into water, the angle of refraction is the angle whose sine is $\frac{3}{4}$ the sine of the angle of incidence. If the zenith distance of the sun is 45°, and is increasing 10° per hour, what is the rate of the angle of refraction? What is its rate just as the sun sets?

CHAPTER IV.

SUCCESSIVE DIFFERENTIALS AND DERIVATIVES.

XII.

THE DIFFERENTIATION OF FUNCTIONS OF THE TIME.

65. When a variable quantity is expressed as a function of t (the time elapsed from a given instant or *origin* of time), its *rate* has a definite value for each instant (Art. 11), and consequently is itself a function of the time. The differentiation of functions of t has not hitherto received especial attention, but the processes already established for the differentiation of functions of x apply likewise to those of t; thus,—

$$d\left(\frac{a}{x}\right) = -\frac{a\,dx}{x^2};$$

so likewise

$$d\left(\frac{a}{t}\right) = -\frac{a\,dt}{t^2}.\text{*}$$

When, however, t denotes the elapsing time, dt is not arbitrary like dx (to which any value may be assigned), but is obviously constant, and has the numerical value unity (for the date of elapsing time increases at the rate of one second per second). It is nevertheless necessary to retain the symbol dt to characterize the expression in which it occurs as a *differential expression* derived from a function of t; this necessity will be more apparent as we proceed.

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* The method we have pursued, of introducing at the outset functions of an independent variable with an *arbitrary* differential, has this advantage: since we are at liberty to substitute for dx any differential whatever, either constant or variable, we can at once differentiate any "function of a function." In order to establish thus generally the formulas for rates, it was necessary to demonstrate the Fundamental Theorem, that the arbitrary differential dx appears only as a factor; thus—

$$d[f(x)] = f'(x).\,dx.$$

SPACE DESCRIBED, AND VELOCITY.

66. The application of the principles of *the Calculus* to functions of the time is of frequent occurrence in Mechanics. Thus, for example, if s denote the space described in the time t by a moving body under the influence of known forces, s will be a function of t, and its derivative $\frac{ds}{dt}$ will also be a function of t. Since $dt = 1$, this derivative will be equal to the rate of the variable s, and hence to the velocity of the moving body (Art. 2). Denoting the velocity by v, we have, therefore,

$$v = \frac{ds}{dt}.$$

67. In the case of a freely falling body, the formula for s is

$$s = \tfrac{1}{2} gt^2,$$

where g is a constant whose value is determined by experiment, and t is the time of falling. Taking the derivative, we have

$$v = \frac{ds}{dt} = gt,$$

the formula for the velocity; these formulas are thus consistent, and each has been experimentally verified.

SECOND DIFFERENTIALS.

68. When the differential of a variable is not constant, it has its own rate of change or differential; this differential of the differential is called the *second differential* of the original variable. In other words, x denoting the variable, if dx is not constant $d(dx)$ will have a value. The symbol d^2x (which may be read *d-second x*) is used as an abbreviation of $d(dx)$.

69. Thus, resuming the functions of t discussed in Art. 9

and Art. 10, if $x = t^2$, it was remarked that dx must be an increasing quantity, hence d^2x should be positive; accordingly we find

$$dx = 2t\,dt \qquad \text{and,} \qquad \text{therefore, } d^2x = 2(dt)^2.$$

Also, in the case of the function $x = \frac{a^2}{t}$, it was remarked that dx should be negative and numerically decreasing; accordingly we have

$$dx = -\frac{a^2dt}{t^2};$$

since dx is negative and decreasing numerically, its *rate* must be regarded as one of increase, and consequently we should expect d^2x to be positive; in fact we find

$$d^2x = \frac{2a^2(dt)^2}{t^3}, \text{ a positive quantity.}$$

Finally, if

$$x = \sin\frac{t\pi}{2},$$

it will *increase* with a decreasing rate until $t = 1$, then *decrease* until $t = 3$; accordingly we find

$$dx = \frac{\pi}{2}\cos\frac{t\pi}{2}\,dt, \qquad \text{and} \qquad d^2x = -\frac{\pi^2}{4}\sin\frac{t\pi}{2}\,.\,(dt)^2.$$

The value of d^2x is clearly negative until t reaches the value 2; dx has then its "greatest negative," or in reality its least value.

In each of the above second differentials the factor $(dt)^2$ presents itself, as dt presented itself in the first differentials, and is retained for a similar reason, although its value is evidently unity.

VELOCITY AND ACCELERATION.

70. The increase per unit of time in the velocity of a moving body is called in Mechanics the *acceleration* of the motion, and the decrease in the velocity per unit of time is called the *retardation* of the motion. The term *acceleration* is, however, in general so used as to include retardation whenever the formulas for determining it give negative values.

From what has been said above, it is evident that acceleration corresponds to a *positive* value of the second differential of the space described, and retardation to a *negative* value.

71. The velocity v of a moving body being a function of t, $\frac{dv}{dt}$ is also a function of t, and since dt is unity, this derivative will be the measure of the rate of v or of the acceleration of the motion. Denoting acceleration by α, we have, therefore,

$$\alpha = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}.$$

$(dt)^2$ is usually written dt^2 when it appears in the denominator of a derivative.

In the case of a freely falling body (Art. 67),

$$s = \tfrac{1}{2}gt^2 \ \therefore \ v = \frac{ds}{dt} = gt, \text{ and } \alpha = \frac{d^2s}{dt^2} = g.$$

The last equation expresses that the acceleration or increase of velocity per second is constant, and equal to the velocity acquired at the end of the first second, or when $t = 1$; a result which has been verified by experiment.

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